

ENERGY AND LAPLACIAN ENERGY ON INVERSE FUZZY GRAPH

R. Keerthana and S. Venkatesh*

Department of Mathematics,
Saveetha Engineering College,
Thandalam, Chennai - 602105, INDIA

E-mail : keerthanar@saveetha.ac.in

*Department of Mathematics
Srinivasa Ramanujan Centre,
SASTRA(Deemed University), Kumbakonam, INDIA

E-mail : venkatesh.s@src.sastra.edu

(**Received:** Dec. 30, 2025 **Accepted:** Apr. 04, 2026 **Published:** Apr. 30, 2026)

Abstract: Real-time problems involving uncertain or indeterminate information can be effectively addressed using fuzzy graphs (FGs). However, the problems in which the edge membership values have a significant impact due to the combination of its corresponding vertices remain unsolved. To address this, inverse fuzzy graph (IFG) was introduced. In this article, the energy (E^I) and the Laplacian energy (LE) on inverse fuzzy graph (IFG) G^I have been newly introduced. Further, various lower and upper bounds are derived for $E^I(G^I)$ and $LE(G^I)$. Additionally the sharp bound is estimated for $E^I(G^I)$ which aids in determining the minimum and maximum bounds for $E^I(G^I)$. Furthermore, for the newly defined l - regular IFG, the equality condition $E^I(G^I) = LE(G^I)$ holds true.

Keywords and Phrases: Inverse fuzzy graph; l - Regular Inverse fuzzy graph; Energy; Laplacian Energy.

2020 Mathematics Subject Classification: 05C72.

1. Introduction

Real-time problems possessing ambiguous information employ Fuzzy graph (FG) theory introduced by Rosenfeld [21]. Due to its efficient approach towards

uncertain problems, FG has been widely applied in data mining, social network, signal processing, decision-making problems, scheduling theory, etc. Mordeson and Chang-Shyh [15] discussed various operations on FG. Following this, the concept of regularity on FG has been implemented by Gani and Radha [9]. Recently, Josy et al. [11] studied neighborhood connectivity index in FG. Further, Islam and Pal [10] proposed second zagreb index in FG with an application in mathematical chemistry.

Table 1: Summary on IFG

Authors	Title
Borzooei, R.A., Almallah, R., Jun, Y.B. and Ghaznavi, H. [7]	Inverse fuzzy graphs with applications
Poulik, S. and Ghorai, G. [17]	New concepts of inverse fuzzy mixed graphs and its application
Borzooei, R.A. and Almallah, R. [8]	Inverse fuzzy multigraphs and planarity with application in decision-making
Almallah, R., Borzooei, R.A. and Jun, Y.B. [2]	Domination numbers of inverse fuzzy graphs with application in decision-making problems
Mondal, R. and Ghorai, G. [14]	Inverse fuzzy mixed planar graphs with application
Keerthana, R., Venkatesh, S., Gobithaasan, R. U. and Salah, J. [13]	Inverse fuzzy-directed graph with an application in traffic flow problem
Keerthana, R. and Venkatesh, S. [12]	A systematic study of intercropping agricultural system using fuzzy and inverse fuzzy hypersoft hypergraph

Graph energy has been extensively applied in the field of molecular chemistry and it is further extended to FG by Anjali and Mathew [3]. The notion of energy in intuitionistic FG was introduced by Praba et al. [18]. Rahimi and Fayazi [19] studied the Laplacian energy in FG and it was further extended by Basha and

Kartheek [5]. Naz et al. [16] defined the energy on bipolar FG with an application in decision-making problems. The energy on pythagorean FG was proposed by Akram and Naz [1]. Recently, Shi et al. [22] investigated various energies on picture FG. Atheequ et al. [4] utilized the Laplacian energy of q -rung orthopair FG for electricity production management. Reddy et al. [20] introduced the Laplacian energy in hesitancy FG. For further study, readers can refer to [6], [23]. There are various real time problems possessing uncertain, indeterminate information can be effectively solved by FGs. However, problems in which edge membership values are significantly influenced by the combination of their corresponding vertices remain unsolved. To address this, inverse fuzzy graph (IFG) was introduced by Borzooei et al. [7] and is summarized in Table 1.

1.1. Motivation and Novelty of the research work

From the literature, we observe that the energy and the Laplacian energy have not been introduced under inverse fuzzy environment. Motivated by this, the energy $E^I(G^I)$ and Laplacian energy $LE(G^I)$ on inverse fuzzy graphs are newly defined along with various lower and upper bounds. Further, l -regular IFG (l -RIFG) is defined and various bounds are evaluated in both $E^I(G^I)$ and $LE(G^I)$. The structure of the paper is organized as outlined, with a focus on the following key points:

1. Introduction of the concept of the energy and Laplacian energy in inverse fuzzy graph.
2. Subsequently, various upper and lower bounds are evaluated for $E^I(G^I)$ and $LE(G^I)$. Further, the sharp bound for $E^I(G^I)$ is also computed.
3. In addition, l -RIFG is newly defined along with the evaluation of various bounds. Also, it is proved that both $E^I(G^I)$ and $LE(G^I)$ are equal in l -RIFG.

2. Preliminaries

Definition 2.1. [21] A fuzzy graph $G = (\sigma, \mu)$ is an ordered pair with underlying crisp graph $G^* = (V^*, E^*)$ along with the membership functions $\sigma : V^* \rightarrow [0, 1]$ and $\mu : V^* \times V^* \rightarrow [0, 1]$ such that $\mu(x, y) \leq \min\{\sigma(x), \sigma(y)\}$ for every edge in G .

Definition 2.2. [7] Consider a simple graph $G^* = (V^*, E^*)$ and $\sigma^I : V^* \rightarrow [0, 1]$ and $\mu^I : E^* \rightarrow [0, 1]$ are two fuzzy subsets on V^* and E^* . An inverse fuzzy graph $G^I = (\sigma^I, \mu^I)$ is defined by $\mu^I(x, y) \geq \min\{\sigma^I(x), \sigma^I(y)\}$, $\forall (x, y) \in E^*$.

Definition 2.3. [7] Let G^I be an inverse fuzzy graph. The degree of a vertex is defined by

$$d^I(x) = \sum_{(x,y) \in E^*} \mu^I(x,y)$$

Definition 2.4. [3] The adjacency matrix A of a fuzzy graph $G = (V^*, \sigma, \mu)$ of order r is an $r \times r$ matrix defined as $A = [a_{ij}]$ where $a_{ij} = \mu(x_i, x_j)$. The eigenvalues of A are called eigenvalues of G . The spectrum of A is called the spectrum of G .

Definition 2.5. [3] Let $G = (V^*, \sigma, \mu)$ be a fuzzy graph and A be its adjacency matrix. The Energy of G is defined as the sum of absolute values of eigenvalues of G .

Definition 2.6. [19] The degree matrix $D = [d_{ij}]$ of a fuzzy graph G of order r is a $r \times r$ diagonal matrix given by

$$d_{ij} = \begin{cases} d(x_i), & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.7. [19] Let A be the adjacency matrix and D be the degree matrix of a fuzzy graph G . Then, $L = D - A$ is called the Laplacian matrix of G .

Definition 2.8. [19] Let G be a fuzzy graph with r vertices and $\Lambda_1 \geq \Lambda_2 \geq \dots \geq \Lambda_r$ be the Laplacian eigenvalues of G . The Laplacian energy of G is defined by

$$LE(G) = \sum_{t=1}^r \left| \Lambda_t - \frac{\left(2 \sum_{1 \leq i < j \leq r} \mu(x_i, x_j) \right)}{r} \right|$$

The comprehensive list of notations used in

Table 2: Comprehensive notation summary for inverse fuzzy graph

Notation	Description
$G^I = (V, \sigma^I, \mu^I)$	Inverse fuzzy graph
σ^I	Vertex membership function
μ^I	Edge membership function
$d^I(u_k)$	Degree of vertex u_k in G^I
l	Average degree or constant used in spectral relation
l -RIFG	l -regular inverse fuzzy graph

A^I	Adjacency matrix of G^I
D^I	Degree matrix of G^I
$L^I = D^I - A^I$	Laplacian matrix
λ_k^I	k -th eigenvalue of A^I
Λ_k^I	k -th eigenvalue of L^I
γ_k^I	k -th eigenvalue of S^I
r	Order of the matrix (number of vertices)
E^I	Energy of inverse fuzzy graph
LE	Laplacian energy
$\sum_{u_k \in V^*} d^I(u_k)$	Sum of degrees (handshaking identity LHS)
$\sum_{(u_k, u_t) \in E^*} \mu^I(u_k, u_t)$	Sum of edge memberships

3. Energy of Inverse Fuzzy Graph

In this section, the energy of G^I is defined and various bounds are discussed for the energy of IFG and l -RIFG. Further, the sharp bounds for $E^I(G^I)$ are also estimated.

Definition 3.1. *The adjacency matrix A^I of IFG G^I of order r is a $r \times r$ matrix of the form $A^I = [a_{kt}^I]$, where $[a_{kt}^I] = \mu^I(\varkappa_k, \varkappa_t)$ and $k, t = 1, 2, \dots, r$.*

Definition 3.2. *The eigenvalues of A^I are the eigenvalues of G^I and denoted by $\lambda_k^I, k = 1, 2, \dots, r$. The spectrum $S^I(G^I)$ of G^I is a set of eigenvalues $\{\lambda_1^I, \lambda_2^I, \dots, \lambda_r^I\}$ of A^I .*

Definition 3.3. *The energy of G^I , $E^I(G^I)$ is the sum of absolute eigenvalues of G^I and is given by $E^I(G^I) = \sum_{k=1}^r |\lambda_k^I|$.*

Example 3.1. For the IFG G^I in Figure 1, the adjacency matrix is given by,

$$A^I(G^I) = \begin{bmatrix} 0 & 0 & 0.7 & 0 & 1 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0.7 & 0.4 & 0 \\ 0.7 & 0 & 0 & 0 & 0 & 0 & 1 & 0.7 \\ 0 & 0.4 & 0 & 0 & 1 & 0 & 0 & 0.7 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0.7 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0.7 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The spectrum $S^I(G^I)$ is $\{1.92594765, 1.01942319, 0.76383715, 0.49790778,$

where $|A|$ is the determinant of $A^I(G^I)$.

Proof. Applying Cauchy-Schwartz inequality to the r numbers $(1, 1, \dots, 1)$ and $(|\lambda_1^I|, |\lambda_2^I|, \dots, |\lambda_r^I|)$,

$$\sum_{k=1}^r |\lambda_k^I| \leq (r)^{\frac{1}{2}} \left(\sum_{k=1}^r (|\lambda_k^I|)^2 \right)^{\frac{1}{2}} \quad (1)$$

$$\left(\sum_{k=1}^r |\lambda_k^I| \right)^2 = \sum_{k=1}^r (|\lambda_k^I|)^2 + 2 \sum_{1 \leq k < t \leq r} \lambda_k^I \lambda_t^I \quad (2)$$

Comparing the coefficients of $(\lambda^I)^{(r-2)}$ in the characteristic polynomial,

$$\prod_{k=1}^r (\lambda^I - \lambda_k^I) = |A^I(G^I) - \lambda^I I|$$

then,

$$\sum_{1 \leq k < t \leq r} \lambda_k^I \lambda_t^I = - \sum_{1 \leq k < t \leq r} (\mu^I(\mathcal{A}_k, \mathcal{A}_t))^2 \quad (3)$$

From the equations (3) and (2),

$$\sum_{k=1}^r (|\lambda_k^I|)^2 = 2 \sum_{1 \leq k < t \leq r} (\mu^I(\mathcal{A}_k, \mathcal{A}_t))^2 \quad (4)$$

From the equations (4) and (1),

$$\sum_{k=1}^r |\lambda_k^I| \leq (r)^{\frac{1}{2}} \left(2 \sum_{1 \leq k < t \leq r} (\mu^I(\mathcal{A}_k, \mathcal{A}_t))^2 \right)^{\frac{1}{2}} = (2r \sum_{1 \leq k < t \leq r} (\mu^I(\mathcal{A}_k, \mathcal{A}_t))^2)^{\frac{1}{2}}.$$

Thus,

$$E^I(G^I) \leq \left(2r \sum_{1 \leq k < t \leq r} (\mu^I(\mathcal{A}_k, \mathcal{A}_t))^2 \right)^{\frac{1}{2}}$$

For any IFG G^I ,

$$\begin{aligned} (E^I(G^I))^2 &= \left(\sum_{k=1}^r |\lambda_k^I| \right)^2 = \sum_{k=1}^r (|\lambda_k^I|)^2 + \sum_{1 \leq k < t \leq r} \lambda_k^I \lambda_t^I \\ &= 2 \sum_{1 \leq k < t \leq r} (\mu^I(\mathcal{A}_k, \mathcal{A}_t))^2 + \frac{2r(r-1)}{2} AM(|\lambda_k^I \lambda_t^I|) \end{aligned}$$

$$AM(|\lambda_k^I \lambda_t^I|) \geq GM(|\lambda_k^I \lambda_t^I|), 1 \leq k < t \leq r$$

$$E^I(G^I) \geq \left(2 \sum_{1 \leq k < t \leq r} (\mu^I(\varkappa_k, \varkappa_t))^2 + r(r-1)GM(|\lambda_k^I \lambda_t^I|) \right)^{\frac{1}{2}}$$

But,

$$GM(|\lambda_k^I \lambda_t^I|) = \left(\prod_{1 \leq k < t \leq r} |\lambda_k^I \lambda_t^I| \right)^{\frac{2}{r(r-1)}}$$

$$= \left(\prod_{k=1}^r (|\lambda_k^I|)^{r-1} \right)^{\frac{2}{r(r-1)}}$$

$$= \left(\prod_{k=1}^r (|\lambda_k^I|) \right)^{\frac{2}{r}} = |A|^{\frac{2}{r}}$$

Thus,

$$E^I(G^I) \geq \left(2 \sum_{1 \leq k < t \leq r} (\mu^I(\varkappa_k, \varkappa_t))^2 + r(r-1)|A|^{\frac{2}{r}} \right)^{\frac{1}{2}}.$$

Definition 3.4. An IFG G^I is said to be l -regular IFG(l -RIFG) if the degree of a vertex, $d^I(\varkappa_k) = l, \forall \varkappa_k \in V^*, k = 1, 2, \dots, r$.

Theorem 3.3. For any IFG G^I , $\sum_{\varkappa_k \in V^*} d^I(\varkappa_k) = 2 \sum_{(\varkappa_k, \varkappa_t) \in E^*} \mu^I(\varkappa_k, \varkappa_t)$.

Proof. By definition, the degree of a vertex $\varkappa_k \in V^*$ in inverse fuzzy graph is given by

$$d^I(\varkappa_k) = \sum_{\varkappa_k \in V^*, \varkappa_k \neq \varkappa_t} \mu^I(\varkappa_k, \varkappa_t)$$

Summing over all vertices, we get

$$\sum_{\varkappa_k \in V^*} d^I(\varkappa_k) = \sum_{\varkappa_k \in V^*} \sum_{\varkappa_k \in V^*, \varkappa_k \neq \varkappa_t} \mu^I(\varkappa_k, \varkappa_t)$$

Since $\mu^I(\varkappa_k, \varkappa_t) = \mu^I(\varkappa_t, \varkappa_k)$ for an undirected IFG, each edge $(\varkappa_t, \varkappa_k)$ contributes twice in the above double summation: once as $\mu^I(\varkappa_k, \varkappa_t)$ and once as $\mu^I(\varkappa_t, \varkappa_k)$. Therefore, $\sum_{\varkappa_k \in V^*} d^I(\varkappa_k) = 2 \sum_{(\varkappa_k, \varkappa_t) \in E^*} \mu^I(\varkappa_k, \varkappa_t)$.

The following theorem yields the sharp bounds for $E^I(G^I)$.

Theorem 3.4. For any IFG G^I with energy $E^I(G^I)$, we have

$$2 \left(\sum_{1 \leq k < t \leq r} (\mu^I(\varkappa_k, \varkappa_t))^2 \right)^{\frac{1}{2}} \leq E^I(G^I) \leq 2 \sum_{1 \leq k < t \leq r} \mu^I(\varkappa_k, \varkappa_t)$$

Proof.

$$\begin{aligned}
 (E^I(G^I))^2 &= \left(\sum_{k=1}^r |\lambda_k^I| \right)^2 \\
 &= \sum_{k=1}^r (|\lambda_k^I|)^2 + 2 \sum_{1 \leq k < t \leq r} |\lambda_k^I| |\lambda_t^I| \\
 &\geq 2 \sum_{1 \leq k < t \leq r} (\mu^I(\varkappa_k, \varkappa_t))^2 + 2 \sum_{1 \leq k < t \leq r} (| - \mu^I(\varkappa_k, \varkappa_t) |)^2 \\
 &= 4 \sum_{1 \leq k < t \leq r} (\mu^I(\varkappa_k, \varkappa_t))^2 \\
 \text{Thus, } E^I(G^I) &\geq 2 \left(\sum_{1 \leq k < t \leq r} (\mu^I(\varkappa_k, \varkappa_t))^2 \right)^{\frac{1}{2}}
 \end{aligned}$$

From the theorem 3.2,

$$\begin{aligned}
 E^I(G^I) &\leq \left(2r \sum_{1 \leq k < t \leq r} (\mu^I(\varkappa_k, \varkappa_t))^2 \right)^{\frac{1}{2}} \\
 &\leq \left(4 \left(\sum_{1 \leq k < t \leq r} \mu^I(\varkappa_k, \varkappa_t) \right)^2 \right)^{\frac{1}{2}} \\
 &= 2 \sum_{1 \leq k < t \leq r} \mu^I(\varkappa_k, \varkappa_t)
 \end{aligned}$$

Hence,

$$\left(2 \sum_{1 \leq k < t \leq r} (\mu^I(\varkappa_k, \varkappa_t))^2 \right)^{\frac{1}{2}} \leq E^I(G^I) \leq 2 \sum_{1 \leq k < t \leq r} \mu^I(\varkappa_k, \varkappa_t)$$

Theorem 3.5. Let G^I be l -RIFG of order r with energy E^I , then $E^I(G^I) \leq lr$.

Proof. By theorem 3.3, for any $\varkappa_k \in V^*$

$$\sum_{\varkappa_k \in V^*} d^I(\varkappa_k) = 2 \sum_{(\varkappa_k, \varkappa_t) \in E^*} \mu^I(\varkappa_k, \varkappa_t)$$

But, $\sum_{\varkappa_k \in V^*} d^I(\varkappa_k) = lr$, then $2 \sum_{(\varkappa_k, \varkappa_t) \in E^*} \mu^I(\varkappa_k, \varkappa_t) = lr$

By theorem 3.4, for any graph G^I ,

$$\begin{aligned}
 E^I(G^I) &\leq 2 \sum_{1 \leq k < t \leq r} \mu^I(\varkappa_k, \varkappa_t) \\
 &= lr.
 \end{aligned}$$

Thus, $E^I(G^I) \leq lr$.

Lemma 3.1. Let G^I be an inverse fuzzy graph such that

$$\sum_{k=1}^r \sigma^I(\varkappa_k) > 1 \quad \text{and} \quad \mu^I(\varkappa_k, \varkappa_t) > \frac{\sigma^I(\varkappa_k) + \sigma^I(\varkappa_t)}{2}$$

for all $(\varkappa_k, \varkappa_t) \in E^*$. Then,

$$\sum_{1 \leq k < t \leq r} (\mu^I(\varkappa_k, \varkappa_t))^2 > \sum_{k=1}^r \sigma^I(\varkappa_k) - 1.$$

Proof. From the hypothesis,

$$\mu^I(\varkappa_k, \varkappa_t) > \frac{\sigma^I(\varkappa_k) + \sigma^I(\varkappa_t)}{2}.$$

Squaring both sides gives

$$(\mu^I(\varkappa_k, \varkappa_t))^2 > \frac{(\sigma^I(\varkappa_k) + \sigma^I(\varkappa_t))^2}{4}.$$

Summing over all $1 \leq k < t \leq r$, we obtain

$$\sum_{k < t} (\mu^I(\varkappa_k, \varkappa_t))^2 > \frac{1}{4} \sum_{k < t} (\sigma^I(\varkappa_k) + \sigma^I(\varkappa_t))^2.$$

Expanding,

$$(\sigma^I(\varkappa_k) + \sigma^I(\varkappa_t))^2 = (\sigma^I(\varkappa_k))^2 + (\sigma^I(\varkappa_t))^2 + 2\sigma^I(\varkappa_k)\sigma^I(\varkappa_t).$$

Thus,

$$\sum_{k < t} (\sigma^I(\varkappa_k) + \sigma^I(\varkappa_t))^2 = (r-1) \sum_{k=1}^r (\sigma^I(\varkappa_k))^2 + 2 \sum_{k < t} \sigma^I(\varkappa_k)\sigma^I(\varkappa_t).$$

Using the identity

$$\sum_{k < t} \sigma^I(\varkappa_k)\sigma^I(\varkappa_t) = \frac{1}{2} \left[\left(\sum_{k=1}^r \sigma^I(\varkappa_k) \right)^2 - \sum_{k=1}^r (\sigma^I(\varkappa_k))^2 \right],$$

we get

$$\sum_{k < t} (\sigma^I(\mathcal{X}_k) + \sigma^I(\mathcal{X}_t))^2 = (r - 2) \sum_{k=1}^r (\sigma^I(\mathcal{X}_k))^2 + \left(\sum_{k=1}^r \sigma^I(\mathcal{X}_k) \right)^2.$$

Hence,

$$\sum_{k < t} (\mu^I(\mathcal{X}_k, \mathcal{X}_t))^2 > \frac{1}{4} \left[(r - 2) \sum_{k=1}^r (\sigma^I(\mathcal{X}_k))^2 + \left(\sum_{k=1}^r \sigma^I(\mathcal{X}_k) \right)^2 \right].$$

Since $\sum_{k=1}^r \sigma^I(\mathcal{X}_k) > 1$, the right-hand side is strictly greater than $\sum_{k=1}^r \sigma^I(\mathcal{X}_k) - 1$. Therefore,

$$\sum_{k < t} (\mu^I(\mathcal{X}_k, \mathcal{X}_t))^2 > \sum_{k=1}^r \sigma^I(\mathcal{X}_k) - 1.$$

Theorem 3.6. *If G^I is IFG without isolated vertices, then*

$$E^I(G^I) \geq 2 \left(\sum_{k=1}^r \sigma^I(\mathcal{X}_k) - 1 \right)^{\frac{1}{2}}$$

provided $\sum_{k=1}^r \sigma^I(\mathcal{X}_k) > 1$ and $\mu^I(\mathcal{X}_k, \mathcal{X}_t) > \frac{\sigma^I(\mathcal{X}_k) + \sigma^I(\mathcal{X}_t)}{2}$

Proof.

Case 1. G^I is connected.

If G^I is connected, $\sum_{k=1}^r \sigma^I(\mathcal{X}_k) > 1$ and $\mu^I(\mathcal{X}_k, \mathcal{X}_t) > \frac{\sigma^I(\mathcal{X}_k) + \sigma^I(\mathcal{X}_t)}{2}$ then, from lemma 3.1, we have By the Lemma, we have

$$\sum_{1 \leq k < t \leq r} (\mu^I(\mathcal{X}_k, \mathcal{X}_t))^2 > \sum_{k=1}^r \sigma^I(\mathcal{X}_k) - 1.$$

From Theorem 3.4,

$$EI(G^I) \geq 2 \left(\sum_{1 \leq k < t \leq r} (\mu^I(\mathcal{X}_k, \mathcal{X}_t))^2 \right)^{1/2}.$$

Therefore,

$$EI(G^I) \geq 2 \left(\sum_{k=1}^r \sigma^I(\mathcal{X}_k) - 1 \right)^{1/2}.$$

Case 2. G^I is disconnected.

Suppose that G^I contains q components $G_1^I, G_2^I, \dots, G_q^I$ having r_1, r_2, \dots, r_q vertices such that $r_1 + r_2 + \dots + r_q = r$, $\sum_{i=1}^q \delta_i \geq 1$, where each $\delta_i = \sum_{k=1}^{r_i} \sigma_{ik}^I$, $i = 1, 2, \dots, q$ and $\sigma_{1k}^I + \sigma_{2k}^I + \dots + \sigma_{qk}^I = \sigma_k^I$ and each component of G^I satisfies membership values of the edges given, we have

$$\begin{aligned}
 E^I(G^I) &\geq 2 \left(\sum_{i=1}^q (\delta_i - 1) \right)^{\frac{1}{2}} \\
 &= 2 \left((\delta_1 - 1)^{\frac{1}{2}} + (\delta_2 - 1)^{\frac{1}{2}} + \dots + (\delta_q - 1)^{\frac{1}{2}} \right) \\
 &= 2 \left(\left(\sum_{k=1}^{r_1} \sigma_{1k}^I - 1 \right)^{\frac{1}{2}} + \left(\sum_{k=1}^{r_2} \sigma_{2k}^I - 1 \right)^{\frac{1}{2}} + \dots + \left(\sum_{k=1}^{r_q} \sigma_{qk}^I - 1 \right)^{\frac{1}{2}} \right) \\
 &= 2 \left(\left(\left(\sum_{k=1}^{r_1} \sigma_{1k}^I - 1 \right)^{\frac{1}{2}} + \left(\sum_{k=1}^{r_2} \sigma_{2k}^I - 1 \right)^{\frac{1}{2}} + \dots + \left(\sum_{k=1}^{r_q} \sigma_{qk}^I - 1 \right)^{\frac{1}{2}} \right)^2 \right)^{\frac{1}{2}} \\
 &= 2 \left(\left(\sum_{k=1}^{r_1} \sigma_{1k}^I - 1 \right)^{\frac{1}{2}} + \left(\sum_{k=1}^{r_2} \sigma_{2k}^I - 1 \right)^{\frac{1}{2}} + \dots + \left(\sum_{k=1}^{r_q} \sigma_{qk}^I - 1 \right)^{\frac{1}{2}} \right) \\
 &\quad + 2 \sum_{i < m} \left(\sum_{k=1}^{r_i} \sigma_{ik}^I - 1 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{r_m} \sigma_{mk}^I - 1 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\
 &\geq 2 \left(\left(\sum_{k=1}^r \sigma^I(\mathcal{A}_k) - 1 \right) + (q - 1)^2 \right)^{\frac{1}{2}}
 \end{aligned}$$

From the previous steps, we obtain

$$EI(G^I) \geq 2 \left(\sum_{k=1}^r \sigma^I(\mathcal{A}_k) - 1 + (q - 1)^2 \right)^{1/2}.$$

Since $(q - 1)^2 \geq 0$, it follows that

$$\begin{aligned}
 \sum_{k=1}^r \sigma^I(\mathcal{A}_k) - 1 + (q - 1)^2 &\geq \sum_{k=1}^r \sigma^I(\mathcal{A}_k) - 1. \\
 \implies \left(\sum_{k=1}^r \sigma^I(\mathcal{A}_k) - 1 + (q - 1)^2 \right)^{1/2} &\geq \left(\sum_{k=1}^r \sigma^I(\mathcal{A}_k) - 1 \right)^{1/2}.
 \end{aligned}$$

Hence,

$$EI(G^I) \geq 2 \left(\sum_{k=1}^r \sigma^I(\varkappa_k) - 1 \right)^{1/2}.$$

From the above, it is immediate to get

$$E^I(G^I) \geq 2 \left(\sum_{k=1}^r \sigma^I(\varkappa_k) - 1 \right)^{\frac{1}{2}}$$

4. Laplacian Energy of Inverse Fuzzy Graph

In this section, various upper and lower bounds are calculated for the Laplacian energy of G^I . Further, for a l -RIFG, it is proved that the $LE(G^I)$ and $E(G^I)$ are equal.

Definition 4.1. The degree matrix $D^I(G^I) = [d_{kt}^I]$ is a $r \times r$ diagonal matrix of the form

$$d_{kt}^I = \begin{cases} d^I(\varkappa_k), & \text{if } k = t \\ 0, & \text{otherwise} \end{cases}$$

Definition 4.2. Let G^I be an IFG. The Laplacian matrix $L^I(G^I)$ is given by $L^I(G^I) = D^I(G^I) - A^I(G^I)$, where $D^I(G^I)$ and $A^I(G^I)$ are respectively the degree matrix and adjacency matrix of G^I . The spectrum $SL^I(G^I)$ are the set of eigenvalues of $L^I(G^I)$.

Definition 4.3. The Laplacian energy of IFG G^I is defined as

$$LE(G^I) = \sum_{p=1}^r \left| \Lambda_p^I - \frac{\left(2 \sum_{1 \leq k < t \leq r} \mu^I(\varkappa_k, \varkappa_t) \right)}{r} \right|$$

where $\Lambda_1^I, \Lambda_2^I, \dots, \Lambda_r^I$ are the Laplacian eigenvalues of G^I .

Definition 4.3. Let G^I be an IFG. Then,

$$M^I = \sum_{1 \leq k < t \leq r} (\mu^I(\varkappa_k, \varkappa_t))^2 + \frac{1}{2} \sum_{k=1}^r \left(d^I(\varkappa_k) - \frac{\left(2 \sum_{1 \leq k < t \leq r} \mu^I(\varkappa_k, \varkappa_t) \right)}{r} \right)^2$$

Example 4.1. For an IFG in Figure 2, the adjacency matrix $A^I(G^I)$ and the degree matrix $D^I(G^I)$ are given by,

$$A^I(G^I) = \begin{bmatrix} 0 & 0.4 & 0.7 & 1 & 1 & 0.4 & 0 & 0 \\ 0.4 & 0 & 1 & 0.4 & 0 & 1 & 0.7 & 0 \\ 0.7 & 1 & 0 & 0.7 & 0 & 0 & 0.4 & 1 \\ 1 & 0.4 & 0.7 & 0 & 0.7 & 0 & 0 & 0.7 \\ 1 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 \\ 0.4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.7 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D^I(G^I) = \begin{bmatrix} 3.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.7 \end{bmatrix}$$

The Laplacian matrix $L^I(G^I)$ is given by,

$$L^I(G^I) = \begin{bmatrix} 3.5 & -0.4 & -0.7 & -1 & -1 & -0.4 & 0 & 0 \\ -0.4 & 3.5 & -1 & -0.4 & 0 & -1 & -0.7 & 0 \\ -0.7 & -1 & 3.8 & -0.7 & 0 & 0 & -0.4 & -1 \\ -1 & -0.4 & -0.7 & 3.5 & -0.7 & 0 & 0 & -0.7 \\ -1 & 0 & 0 & -0.7 & 1.7 & 0 & 0 & 0 \\ -0.4 & -1 & 0 & 0 & 0 & 1.4 & 0 & 0 \\ 0 & -0.7 & -0.4 & 0 & 0 & 0 & 1.1 & 0 \\ 0 & 0 & -1 & -0.7 & 0 & 0 & 0 & 1.7 \end{bmatrix}$$

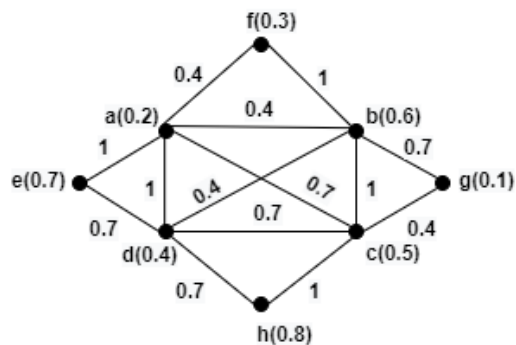


Figure 2: Inverse fuzzy graph G^I

Then $SL^I(G^I) = \{5.074, 4.553, 4.065, 3.020, 1.418, 1.167, 0.899, 0\}$ and the Laplacian energy $LE(G^I) = 13.228$.

Theorem 4.1. *If $\Lambda_1^I \geq \Lambda_2^I \geq \dots \geq \Lambda_r^I$ are the Laplacian eigenvalues of G^I , then*

1. $\sum_{k=1}^r \Lambda_k^I = 2 \sum_{1 \leq k < t \leq r} \mu^I(\mathcal{X}_k, \mathcal{X}_t)$
2. $\sum_{k=1}^r (\Lambda_k^I)^2 = 2 \sum_{1 \leq k < t \leq r} (\mu^I(\mathcal{X}_k, \mathcal{X}_t))^2 + \sum_{k=1}^r (d^I(\mathcal{X}_k))^2$

Proof.

1. Since $L^I(G^I)$ is symmetric having non-negative Laplacian eigenvalues then,

$$\sum_{k=1}^r (\Lambda_k^I) = \text{tr}(L^I) = \sum_{k=1}^r d^I(\mathcal{X}_k) = 2 \sum_{1 \leq k < t \leq r} \mu^I(\mathcal{X}_k, \mathcal{X}_t)$$

$$2. \text{ From } L^I(G^I), \text{ we have } \begin{vmatrix} d^I(\mathcal{X}_1) & \dots & -\mu^I(\mathcal{X}_1 \mathcal{X}_r) \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ -\mu^I(\mathcal{X}_r \mathcal{X}_1) & \dots & d^I(\mathcal{X}_r) \end{vmatrix}$$

Then,

$$\begin{aligned} \text{tr}(L^I)^2 &= ((d^I(\mathcal{X}_1))^2 + (\mu^I(\mathcal{X}_1 \mathcal{X}_2))^2 + (\mu^I(\mathcal{X}_1 \mathcal{X}_3))^2 + \dots + (\mu^I(\mathcal{X}_1 \mathcal{X}_r))^2) \\ &+ ((\mu^I(\mathcal{X}_2 \mathcal{X}_1))^2 + (d^I(\mathcal{X}_2))^2 + (\mu^I(\mathcal{X}_2 \mathcal{X}_3))^2 + \dots + (\mu^I(\mathcal{X}_2 \mathcal{X}_r))^2) \\ &+ \dots + ((\mu^I(\mathcal{X}_r \mathcal{X}_1))^2 + (\mu^I(\mathcal{X}_r \mathcal{X}_2))^2 + (\mu^I(\mathcal{X}_r \mathcal{X}_3))^2 + \dots + (d^I(\mathcal{X}_r))^2) \\ &= 2 \sum_{1 \leq k < t \leq r} (\mu^I(\mathcal{X}_k, \mathcal{X}_t))^2 + \sum_{k=1}^r (d^I(\mathcal{X}_k))^2 \end{aligned}$$

Theorem 4.2. *Let $\Lambda_1^I \geq \Lambda_2^I \geq \dots \geq \Lambda_r^I$ be the Laplacian eigenvalues of G^I and*

$$\gamma_k^I = \left| \Lambda_k^I - \frac{\left(2 \sum_{1 \leq k < t \leq r} \mu^I(\mathcal{X}_k, \mathcal{X}_t) \right)}{r} \right|, \text{ then}$$

1. $\sum_{k=1}^r \gamma_k^I = 0$
2. $\sum_{k=1}^r (\gamma_k^I)^2 = 2M^I$

$$\text{where } M^I = \sum_{1 \leq k < t \leq r} (\mu^I(\mathcal{X}_k, \mathcal{X}_t))^2 + \frac{1}{2} \sum_{k=1}^r \left(d^I(\mathcal{X}_k) - \frac{\left(2 \sum_{1 \leq k < t \leq r} \mu^I(\mathcal{X}_k, \mathcal{X}_t) \right)}{r} \right)^2.$$

Theorem 4.3. *Let $\Lambda_1^I \geq \Lambda_2^I \geq \dots \geq \Lambda_r^I$ be the Laplacian eigenvalues of G^I , then*

$$LE(G^I) \leq \left(2r \sum_{1 \leq k < t \leq r} (\mu^I(\mathcal{X}_k, \mathcal{X}_t))^2 + \frac{1}{2} \sum_{k=1}^r \left(d^I(\mathcal{X}_k) - \frac{(2 \sum_{1 \leq k < t \leq r} \mu^I(\mathcal{X}_k, \mathcal{X}_t))}{r} \right)^2 \right)^{\frac{1}{2}}$$

Proof. Apply Cauchy-Schwartz inequality to $(1, 1, \dots, 1)$ and $(|\gamma_1^I|, |\gamma_2^I|, \dots, |\gamma_r^I|)$, then

$$\left| \sum_{k=1}^r \gamma_k^I \right|^2 \leq r \sum_{k=1}^r |\gamma_k^I|^2$$

where

$$LE(G^I) \leq \left(r \sum_{k=1}^r |\gamma_k^I|^2 \right)^{\frac{1}{2}} = (2M^I r)^{\frac{1}{2}}$$

and $M^I = \sum_{1 \leq k < t \leq r} (\mu^I(\mathcal{X}_k, \mathcal{X}_t))^2 + \frac{1}{2} \sum_{k=1}^r \left(d^I(\mathcal{X}_k) - \frac{(2 \sum_{1 \leq k < t \leq r} \mu^I(\mathcal{X}_k, \mathcal{X}_t))}{r} \right)^2$ Then,

$$LE(G^I) \leq \left(2r \sum_{1 \leq k < t \leq r} (\mu^I(\mathcal{X}_k, \mathcal{X}_t))^2 + \frac{1}{2} \sum_{k=1}^r \left(d^I(\mathcal{X}_k) - \frac{(2 \sum_{1 \leq k < t \leq r} \mu^I(\mathcal{X}_k, \mathcal{X}_t))}{r} \right)^2 \right)^{\frac{1}{2}}$$

Theorem 4.4. Let G^I be an IFG and $\Lambda_1^I \geq \Lambda_2^I \geq \dots \geq \Lambda_r^I$ be the Laplacian eigenvalues of G^I , then

$$LE(G^I) \geq 2 \left(\sum_{1 \leq k < t \leq r} (\mu^I(\mathcal{X}_k, \mathcal{X}_t))^2 + \frac{1}{2} \sum_{k=1}^r \left(d^I(\mathcal{X}_k) - \frac{(2 \sum_{1 \leq k < t \leq r} \mu^I(\mathcal{X}_k, \mathcal{X}_t))}{r} \right)^2 \right)^{\frac{1}{2}}$$

Proof. From the definition of Laplacian energy, we have

$$\begin{aligned} (LE(G^I))^2 &= \left(\sum_{k=1}^r |\gamma_k^I| \right)^2 \\ &= \sum_{k=1}^r |\gamma_k^I|^2 + 2 \sum_{1 \leq k < t \leq r} |\gamma_k^I| |\gamma_t^I| \\ &\geq 4M^I \end{aligned}$$

Then,

$$LE(G^I) \geq 2 \left(\sum_{1 \leq k < t \leq r} (\mu^I(\mathcal{X}_k, \mathcal{X}_t))^2 + \frac{1}{2} \sum_{k=1}^r \left(d^I(\mathcal{X}_k) - \frac{(2 \sum_{1 \leq k < t \leq r} \mu^I(\mathcal{X}_k, \mathcal{X}_t))}{r} \right)^2 \right)^{\frac{1}{2}}$$

Theorem 4.5. Let G^I be an IFG and $\Lambda_1^I \geq \Lambda_2^I \geq \dots \geq \Lambda_r^I$ be the Laplacian eigenvalues of G^I , then

$$LE(G^I) \leq \gamma_1^I + \left((r-1) \left(2 \sum_{1 \leq k < t \leq r} (\mu^I(\mathcal{X}_k, \mathcal{X}_t))^2 + \sum_{k=1}^r \left(d^I(\mathcal{X}_k) - \frac{(2 \sum_{1 \leq k < t \leq r} \mu^I(\mathcal{X}_k, \mathcal{X}_t))}{r} \right)^2 - (\gamma_1^I)^2 \right) \right)^{\frac{1}{2}}$$

Proof. Apply Cauchy-Schwartz inequality to $(1, 1, \dots, 1)$ and $(|\gamma_2^I|, |\gamma_3^I|, \dots, |\gamma_r^I|)$, then

$$\left| \sum_{k=2}^r \gamma_k^I \right|^2 \leq (r-1) \sum_{k=2}^r |\gamma_k^I|^2$$

$$LE(G^I) - \gamma_1^I \leq \left((r-1)(2M^I - (\gamma_1^I)^2) \right)^{\frac{1}{2}}$$

But, $M^I = \sum_{1 \leq k < t \leq r} (\mu^I(\mathcal{X}_k, \mathcal{X}_t))^2 + \frac{1}{2} \sum_{k=1}^r \left(d^I(\mathcal{X}_k) - \frac{(2 \sum_{1 \leq k < t \leq r} \mu^I(\mathcal{X}_k, \mathcal{X}_t))}{r} \right)^2$ Then,

$$LE(G^I) \leq \gamma_1^I + \left((r-1) \left(2 \sum_{1 \leq k < t \leq r} (\mu^I(\mathcal{X}_k, \mathcal{X}_t))^2 + \sum_{k=1}^r \left(d^I(\mathcal{X}_k) - \frac{(2 \sum_{1 \leq k < t \leq r} \mu^I(\mathcal{X}_k, \mathcal{X}_t))}{r} \right)^2 - (\gamma_1^I)^2 \right) \right)^{\frac{1}{2}}$$

Theorem 4.6. Let G^I be a l -RIFG and $\Lambda_1^I \geq \Lambda_2^I \geq \dots \geq \Lambda_r^I$ be the Laplacian eigenvalues of G^I , then

$$LE(G^I) \leq \gamma_1^I + \left((r-1) \left(2 \sum_{1 \leq k < t \leq r} (\mu^I(\mathcal{X}_k, \mathcal{X}_t))^2 - (\gamma_1^I)^2 \right) \right)^{\frac{1}{2}}$$

Proof. Since G^I is a l -RIFG, then

$$l = d^I(\mathcal{X}_k) = \frac{2 \sum_{1 \leq k < t \leq r} \mu^I(\mathcal{X}_k, \mathcal{X}_t)}{r} \quad (5)$$

Using (5) in theorem 4.5, we get

$$LE(G^I) \leq \gamma_1^I + \left((r-1) \left(2 \sum_{1 \leq k < t \leq r} (\mu^I(\mathcal{X}_k, \mathcal{X}_t))^2 - (\gamma_1^I)^2 \right) \right)^{\frac{1}{2}}$$

Theorem 4.7. *If G^I is l -RIFG, then $LE(G^I) = E(G^I)$.*

Proof. Let G^I be an l - RIFG of order r . Then, by Definition 3.4, the degree of each vertex is constant, i.e., $d^I(\varkappa_k) = l, \forall \varkappa_k \in V^*$. Hence, the degree matrix satisfies $D^I(G^I) = lI$. From Definition 4.2, the Laplacian matrix is given by $L^I(G^I) = D^I(G^I) - A^I(G^I) = lI - A^I(G^I)$. Let λ_k^I be an eigenvalue of the adjacency matrix $A^I(G^I)$ with corresponding eigenvector v , i.e., $A^I(G^I)v = \lambda_k^I v$. Then, $L^I(G^I)v = (lI - A^I(G^I))v = lv - \lambda_k^I v = (l - \lambda_k^I)v$. Thus, the Laplacian eigenvalues are given by $\Lambda_k^I = l - \lambda_{r-k+1}$. Since G^I is a l -RIFG, then

$$l = \frac{2 \sum_{1 \leq k < t \leq r} \mu^I(\varkappa_k, \varkappa_t)}{r}$$

and

$$\begin{aligned} \gamma_k^I - \frac{2 \sum_{1 \leq k < t \leq r} \mu^I(\varkappa_k, \varkappa_t)}{r} &= -\lambda_{r-k+1}, \quad k = 1, 2, \dots, r \\ \left| \gamma_k^I - \frac{2 \sum_{1 \leq k < t \leq r} \mu^I(\varkappa_k, \varkappa_t)}{r} \right| &= \left| -\lambda_{r-k+1} \right| \\ \sum_{k=1}^r \left| \gamma_k^I - \frac{2 \sum_{1 \leq k < t \leq r} \mu^I(\varkappa_k, \varkappa_t)}{r} \right| &= \sum_{k=1}^r \left| -\lambda_{r-k+1} \right| \end{aligned}$$

Thus, $LE(G^I) = E(G^I)$.

Example 4.2. For the l - RIFG in Figure 3, $S^I(G^I) = \{1.4, -1, -0.4, 0\}$ and $E^I(G^I) = 2.8$. Also, $SL^I(G^I) = \{2.4, 1.8, 1.4, 0\}$ and $LE(G^I) = 2.8$. Thus, $LE(G^I) = E(G^I)$. From the above theorem, we have, $\Lambda_k^I = l - \lambda_{r-k+1}$. For the first Laplacian eigenvalue $\Lambda_1^I = 2.4$ and eigenvalue $\lambda_1^I = 1.4$, we have $\Lambda_1^I = 2.4 = 1.4 - (-1.0)$ holds true for $k = 1$. Similarly, for $k = 2, \Lambda_2^I = 1.8 = 1.4 - (-0.4)$, for $k = 3, \Lambda_3^I = 1.4 = 1.4 - 0$, for $k = 4, \Lambda_4^I = 0 = 1.4 - 1.4$.

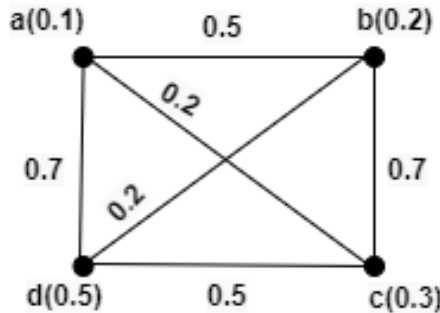


Figure 3: l - RIFG

5. Conclusion

In this article, the notion of energy (E^I) and Laplacian energy (LE) for the IFG G^I have been newly introduced and various upper and lower bounds are derived. Also, the sharp bound for the $E^I(G^I)$ is newly evaluated. Furthermore, l -regular IFG is newly defined and $E^I(G^I) = LE(G^I)$ is true only if G^I is l -regular. The IFG is new in the area of fuzzy graph theory, there are various energy related concepts, Zagreb energy, Randic energy can be further explored on IFG for future work.

References

- [1] Akram, M. and Naz, S., Energy of Pythagorean fuzzy graphs with applications, *Mathematics*, Vol. 6, No. 8 (2018), 136.
- [2] Almallah, R., Borzooei, R. A. and Jun, Y. B., Domination numbers of inverse fuzzy graphs with application in decision-making problems, *New Mathematics and Natural Computation*, Vol. 18, No. 01 (2022), 19-42.
- [3] Anjali, N. and Mathew, S., Energy of a fuzzy graph, *Annals of fuzzy mathematics and Informatics*, Vol. 6, No. 3 (2013), 455-465.
- [4] Atheeque A, M., Basha S, S., Pratyusha, N., Reddy, C. R., Alam, M. N., Ahmad, H. and Tarakaramu, N., The application of cosine similarity measures with Laplacian energy to q-rung orthopair fuzzy graphs in decision-making problems, *AIP Advances*, Vol. 14, No. 5 (2024), 1-12.
- [5] Basha, S. S. and Kartheek, E., Laplacian energy of an intuitionistic fuzzy graph, *Indian Journal of Science and Technology*, Vol. 8, No. 33 (2015), 1-7.
- [6] Bathusha, S. S., Jayakumar, S. and Raj, S. A. K., The Energy of Interval-Valued Complex Neutrosophic Graph Structures: Framework, Application and Future Research Directions, *Neutrosophic Systems with Applications*, Vol. 13, No. 1 (2024), 67-101.
- [7] Borzooei, R. A., Almallah, R., Jun, Y. B. and Ghaznavi, H., Inverse fuzzy graphs with applications, *New Mathematics and Natural Computation*, Vol. 16, No. 02 (2020), 397-418.
- [8] Borzooei, R. A. and Almallah, R., Inverse fuzzy multigraphs and planarity with application in decision-making, *Soft Computing*, Vol. 26, No. 4 (2022), 1531-1539.

- [9] Gani, A. N. and Radha, K., On regular fuzzy graphs, *Journal of Physical sciences*, Vol. 12, No. 2 (2008), 33–40.
- [10] Islam, S. R. and Pal, M., Second Zagreb index for fuzzy graphs and its application in mathematical chemistry, *Iranian Journal of Fuzzy Systems*, Vol. 20, No.1 (2023), 119-136.
- [11] Josy, A., Mathew, S. and Mordeson, J. N., Neighborhood connectivity index of a fuzzy graph and its application to human trafficking, *Iranian Journal of Fuzzy Systems*, Vol. 19, No. 3 (2022), 139-154.
- [12] Keerthana, R. and Venkatesh, S., A systematic study of intercropping agricultural system using fuzzy and inverse fuzzy hypersoft hypergraph, *Journal of Applied Mathematics and Computing*, Vol. 71, No. 3 (2025), 4067-4094.
- [13] Keerthana, R., Venkatesh, S., Gobithaasan, R. U. and Salah, J., Inverse Fuzzy-Directed Graph With an Application in Traffic Flow Problem, *Journal of Mathematics*, Vol. 2024, No. 1 (2024), 1-19.
- [14] Mondal, R. and Ghorai, G., Inverse fuzzy mixed planar graphs with application, *International Journal of Applied and Computational Mathematics*, Vol. 10, No. 4 (2024), 131.
- [15] Mordeson, J. N. and Chang-Shyh, P., Operations on fuzzy graphs, *Information sciences*, Vol. 79, No. (3-4) (1994), 159-170.
- [16] Naz, S., Ashraf, S. and Karaaslan, F., Energy of a bipolar fuzzy graph and its application in decision making, *Italian Journal of Pure and Applied Mathematics*, Vol. 40, No. 1 (2018), 339-352.
- [17] Poulik, S. and Ghorai, G., New concepts of inverse fuzzy mixed graphs and its application, *Granular Computing*, Vol. 7, No. 1 (2021), 1-11.
- [18] Praba, B., Chandrasekaran, V. M. and Deepa, G., Energy of an intuitionistic fuzzy graph, *Ital. J. Pure Appl. Math*, Vol. 32, No. 1 (2014), 431-444.
- [19] Rahimi, S. S. and Fayazi, F., Laplacian energy of a fuzzy graph, *Iranian Journal of Mathematical Chemistry*, Vol. 5, No. 1 (2014), 1-10.
- [20] Reddy, N. R., Khan, M. Z., Basha, S. S., Alahmadi, A., Alahmadi, A. H. and Chowdhary, C. L., The Laplacian energy of hesitancy fuzzy graphs in decision-making problems, *Comput. Syst. Sci. Eng*, Vol. 44, No. 1 (2023), 2637-2653.

- [21] Rosenfeld, A., Fuzzy graphs, In Fuzzy sets and their applications to cognitive and decision processes, Academic press, (1975), 77-95.
- [22] Shi, X., Kosari, S., Talebi, A. A., Sadati, S. H. and Rashmanlou, H., Investigation of the main energies of picture fuzzy graph and its applications, International Journal of Computational Intelligence Systems, Vol. 15, No. 1 (2022), 31.
- [23] Srinivasan, S. and Shanmugavel, V., Minimum Degree Energy of Graphs and Pebbling Graphs, Indian Journal of Science and Technology, 17(18) (2024), 1838-1844.

This page intentionally left blank.